TRINITY COLLEGE

Mathematics Specialist Units 1,2
Test 6 2018

Proof, Complex Numbers

STUDENT'S NAME ______________________________

DATE: Monday 17 September **TIME:** 50 minutes **MARKS:** 52

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, notes on one side of a single A4 page

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

Express the recurring decimal 21.357575757..... as a rational number.

$x = 21.3\overline{57}$

$10x = 213.\overline{57}$

$1000x = 21357.\overline{57}$

$990x = 21144$

$x = \dfrac{21144}{990}$

2. (6 marks)

(a) Given $(PQ)^3 = I$, show $QPQ = P^{-1}Q^{-1}P^{-1}$ where I is the identity matrix, P and Q are non-singular square matrices. [2]

$$PQPQPQ = I$$
$$QPQPQ = P^{-1}$$
$$PQPQ = Q^{-1}P^{-1}$$
$$QPQ = P^{-1}Q^{-1}P^{-1}$$

(b) If matrix A is such that $A^2 = 4A - 7I$ where I is the identity matrix. Express A^4 in the form $pA + qI$. [4]

$$A^4 = A^2A^2$$
$$= (4A - 7I)(4A - 7I)$$
$$= 16A^2 - 28A - 28A + 49I$$
$$= 16(4A - 7I) - 56A + 49I$$
$$= 64A - 112I - 56A + 49I$$
$$= 8A - 63I$$

3. (4 marks)

Determine two numbers which have a sum of 3 and a product of 3.

$$A + B = 3$$
$$B = 3 - A$$

$$AB = 3$$
$$A(3 - A) = 3$$
$$3A - A^2 = 3$$
$$0 = A^2 - 3A + 3$$
$$A = \frac{3 \pm \sqrt{9 - 12}}{2}$$
$$= \frac{3 \pm \sqrt{-3}}{2}$$
$$= \frac{3 \pm i\sqrt{3}}{2}$$

4. (8 marks)

(a) Prove, by contradiction, $\log_{10} 2$ is irrational. [4]

ASSUME $\log_{10} 2$ RATIONAL

ie $\log_{10} 2 = \frac{a}{b}$ WHERE $\frac{a}{b}$ IS SIMPLIEST FORM

$2 = 10^{\frac{a}{b}}$

$2^b = 10^a$

$2^b = 2^a 5^a$

2^b CANNOT HAVE A FACTOR OF 5

$\therefore$ CONTRADICTION OF $\log_{10} 2$ RATIONAL

$\therefore \log_{10} 2$ IRRATIONAL

(b) Prove, by exhaustion, $(n+1)^3 \geq 3^n$ where n is a counting number ≤ 4. [4]

$n = 1$ $\quad 2^3 > 3^1$ TRUE

$n = 2$ $\quad 3^3 > 3^2$ TRUE

$n = 3$ $\quad 4^3 > 3^3$ TRUE

$n = 4$ $\quad 5^3 > 3^4$ TRUE

$\therefore (n+1)^3 \geqslant 3^n \quad n \leq 4$

5. (5 marks)

Prove, by mathematical induction, that $n^3 + 2n$ is divisible by 3 for any positive integer n .

LET $n = 1$ $1^3 + 2(1) = 3$ $\therefore$ DIVISIBLE BY 3

ASSUME TRUE FOR $n = k$

ie $k^3 + 2k = 3m$

PROVE TRUE FOR $n = k+1$

ie PROVE $(k+1)^3 + 2(k+1)$ DIVISIBLE BY 3

$$(k+1)^3 + 2(k+1)$$
$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= k^3 + 2k + 3k^2 + 3k + 3$$
$$= 3m + 3(k^2 + k + 1)$$

= MULTIPLE OF 3.

SINCE TRUE FOR $n = 1$ TRUE FOR $n = 2$

SINCE TRUE FOR $n = 2$ TRUE FOR $n = 3$

AND SO ON.

$\therefore$ $n^3 + 2n$ DIVISIBLE BY 3 FOR $n \geq 1$, $n \in \mathbb{Z}$

6. (7 marks)

Simplify the following complex expressions leaving the answer in the form a + bi.

(a) $6-7i-(2-4i)$ $= 4 - 3i$ [2]

(b) $\frac{4+3i}{1-2i} \times \frac{1+2i}{1+2i}$ [3]

$= \frac{4+8i+3i-6}{1+4}$

$= \frac{-2}{5} + \frac{11i}{5}$

(c) $\frac{-i}{i^3} = \frac{-i}{i\,i^2}$ [2]

$= \frac{-1}{i^2}$

$= 1$

7. (8 marks)

(a) One root of the equation $z^2 + az + b = 0$, where a and b are real constants, is $4 - i$. Determine the value of a and b. [4]

$$(z-(4-i))(z-(4+i))$$

$$= (z-4+i)(z-4-i)$$

$$= z^2 - 4z - \cancel{zi} - 4z + 16 + \cancel{4i} + \cancel{iz} - \cancel{4i} + 1$$

$$= z^2 - 8z + 17$$

$a = -8$

$b = 17$

(b) Solve the equation $3z = (7+2i)^2 - \bar{z}$ for the complex number z. (Hint: let $z = a + bi$) [4]

$$3z + \bar{z} = (7+2i)(7+2i)$$

$$3(a+bi) + a - bi = 49 + 14i + 14i - 4$$

$$4a + 2bi = 45 + 28i$$

Re $\quad 4a = 45$

$\quad a = \frac{45}{4}$

Im $\quad 2b = 28$

$\quad b = 14$

$$z = \frac{45}{4} + 14i$$

8. (4 marks)

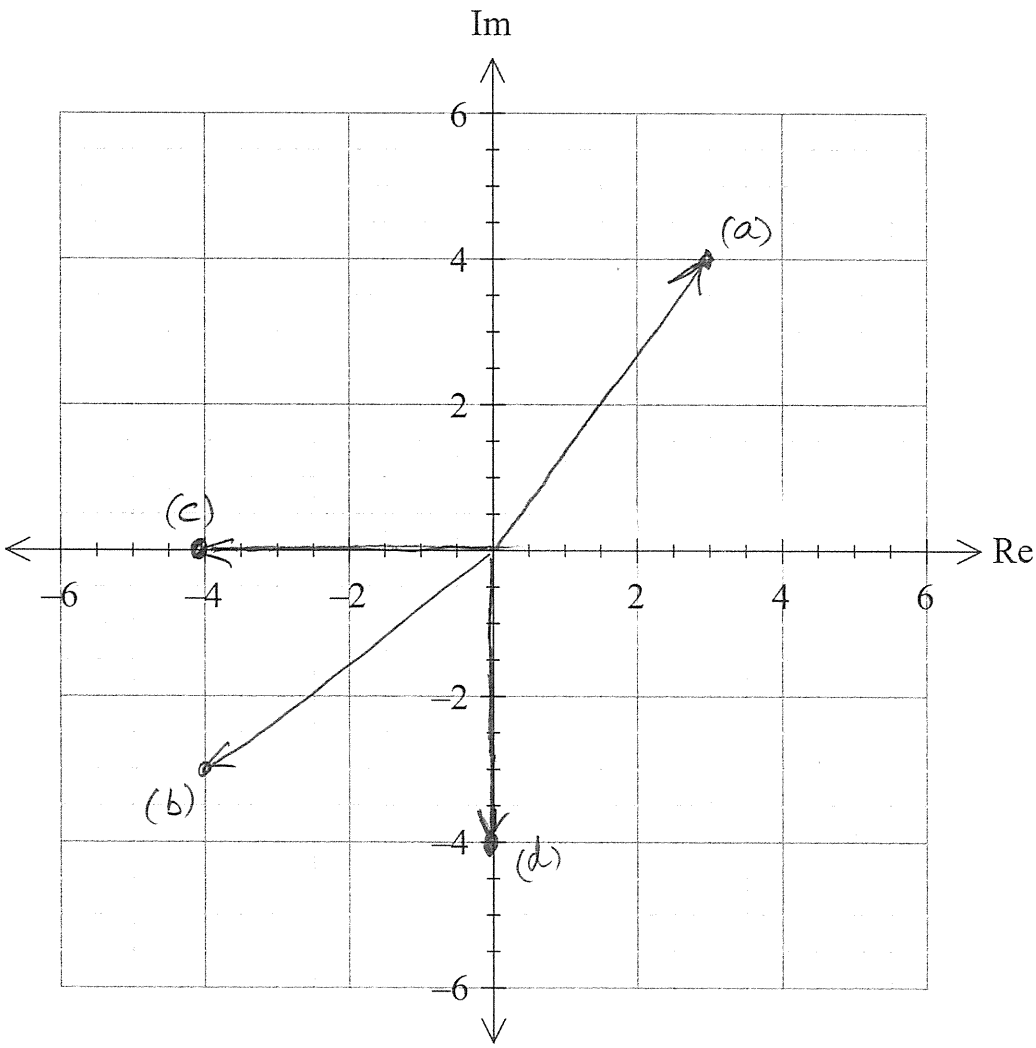
Given $z = 3 - 4i$, draw each of the following on the Argand diagram below. Clearly label each answer.

(a) $\bar{z}$ [1]

(b) $i^3 z$ [1]

(c) Im(z) [1]

(d) $i\,\mathrm{Re}(z)$ [1]



9. (~~6~~ 7 marks)

Use mathematical induction to prove the following conjecture:

$1 + (1+x) + (1+x)^2 + ... + (1+x)^{n-1} = \frac{(1+x)^n - 1}{x}$, $n \geq 1$, n a counting number.

LET $n = 1$ $\quad 1 = \frac{1+x-1}{x}$

$1 = 1$ TRUE

ASSUME TRUE FOR $n = k$

ie $1 + (1+x) + (1+x)^2 + ... + (1+x)^{k-1} = \frac{(1+x)^k - 1}{x}$

PROVE TRUE FOR $n = k+1$

ie PROVE $1 + (1+x) + (1+x)^2 + .. + (1+x)^{k-1} + (1+x)^k = \frac{(1+x)^{k+1} - 1}{x}$

$$LHS = 1 + (1+x) + (1+x)^2 + ... + (1+x)^{k-1} + (1+x)^k$$

$$= \frac{(1+x)^k - 1}{x} + (1+x)^k$$

$$= \frac{(1+x)^k}{x} - \frac{1}{x} + \frac{x(1+x)^k}{x}$$

$$= \frac{(1+x)^k}{x}\left[1 + x\right] - \frac{1}{x}$$

$$= \frac{(1+x)^{k+1}}{x} - \frac{1}{x}$$

$$= \frac{(1+x)^{k+1} - 1}{x}$$

$= RHS$

SINCE TRUE FOR $n = 1$ TRUE FOR $n = 2$

SINCE TRUE FOR $n = 2$ TRUE FOR $n = 3$

AND SO ON

$\therefore 1 + (1+x) + (1+x)^2 + ... (1+x)^{n-1} = \frac{(1+x)^n - 1}{x}$ $\quad n \geq 1$ n COUNTING NUMBER